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Periodical solution of nonlinear vibrating systems with n-DOF

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This contribution deals with a specific approach to the solution of periodical steady state response (if it exists) of a nonlinear system to periodical excitation. It is supposed that the nonlinear force vector depends on a vector of generalized displacements only. Let us assume the equation of motion in the following form:

$$\mathbf{M}_0\ddot{\mathbf{q}}(t) + \mathbf{B}_0\dot{\mathbf{q}}(t) + \mathbf{K}_0\mathbf{q}(t) + \boldsymbol{\xi}\left[\mathbf{q}(t)\right] = \mathbf{f}(t), \tag{1}$$

where $\mathbf{M}_0$, $\mathbf{B}_0$, $\mathbf{K}_0 \in \mathbb{R}^{n,n}$ are the matrices of mass, external damping and gyroscopic effect, stiffness and inner damping, respectively. The remaining symbols

$$\mathbf{f}(t+T) = \mathbf{f}(t), \qquad \mathbf{q}(t) = \mathbf{q}(t+T), \qquad T = \frac{2\pi}{\omega} \tag{2}$$

represent the periodical generalized force vector, the periodical generalized displacement vector and the period depending on the angular speed of excitation ω. The periodical solution of (1) can be expressed by means of matrix periodic Green's function (PGF). This matrix function corresponds to the response of linear part of the mathematical model (1) to Dirac train forcing function acting sequentially in individual DOFs. It means response to the columns of the excitation matrix

$$\boldsymbol{\Delta}_T(t) = \begin{bmatrix} \delta_T(t) & & & \\ & \delta_T(t) & & \\ & & \ddots & \\ & & & \delta_T(t) \end{bmatrix} = \frac{1}{T}\,\mathbf{I}\sum_{k=-\infty}^{\infty} e_k(t), \tag{3}$$

where

$$\delta_T(t) = \sum_{k=-\infty}^{\infty}\delta(t-kT) = \frac{1}{T}\sum_{k=-\infty}^{\infty} e_k(t), \quad e_k(t) = e^{ik\omega t}, \tag{4}$$

and $\mathbf{I}$ is the identity matrix. The PGF function has the form

$$\mathbf{H}_T(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty}\mathbf{L}_k e_k(t), \qquad \mathbf{L}_k = \left(-k^2\omega^2\mathbf{M}_0 + ik\omega\mathbf{B}_0 + \mathbf{K}_0\right)^{-1} \in \mathbb{C}^{n,n}. \tag{5}$$

This function and its development were presented in the preceding papers [1, 2]. The total periodical response has the form

$$\mathbf{q}(t) = \int_0^T \mathbf{H}_T(t-s)\mathbf{f}(s)\,\mathrm{d}s - \int_0^T \mathbf{H}_T(t-s)\boldsymbol{\xi}\left[\mathbf{q}(s)\right]\,\mathrm{d}s. \tag{6}$$

The nonlinear force term is taken into account as a part of excitation and transferred to the right hand side of (1). The first term on the right hand side of (6) can be exactly expressed in the form

$$\mathbf{I}_1 = \int_0^T \mathbf{H}_T(t-s)\mathbf{f}(s)\,\mathrm{d}s = \frac{1}{T}\int_0^T \sum_k \sum_j \mathbf{L}_k\,\mathbf{f}_j\,e_k(t)\,e_{j-k}(s)\,\mathrm{d}s = \sum_k \mathbf{L}_k\,\mathbf{f}_k\,e_k(t). \tag{7}$$

The nonlinear force written as a sequence of Dirac impulses can used for the collocation method

$$\boldsymbol{\xi}\left[\mathbf{q}(t)\right] = \sum_{j=1}^{m} \boldsymbol{\xi}\left[\mathbf{q}(t)\right]\delta(t-t_j) = \sum_{j=1}^{m} \boldsymbol{\xi}\left[\mathbf{q}(t_j)\right]\delta(t-t_j) = \sum_{j=1}^{m} \boldsymbol{\xi}\left[\mathbf{q}(t_j)\right]\Delta t. \tag{8}$$

The second integral takes the form

$$\mathbf{I}_2 = -\int_0^T \mathbf{H}_T(t-s)\boldsymbol{\xi}\left[\mathbf{q}(s)\right]\,\mathrm{d}s = -\frac{\Delta t}{T}\sum_{k=-N}^{N} \mathbf{L}_k\,e_k(t)\sum_{l=1}^{m} e_{-k}(t_l)\boldsymbol{\xi}\left[\mathbf{q}(t_l)\right]. \tag{9}$$

The periodic response can be expressed in implicit form

$$\mathbf{q}(t) = \mathbf{I}_1 + \mathbf{I}_2 = \sum_k \mathbf{L}_k\,\mathbf{f}_k\,e_k(t) - \frac{\Delta t}{T}\sum_{k=-N}^{N} \mathbf{L}_k\,e_k(t)\sum_{l=1}^{m} e_{-k}(t_l)\boldsymbol{\xi}\left[\mathbf{q}(t_l)\right], \tag{10}$$

whose numerical solution can lead to the displacement vector $\mathbf{q}(t_l)$ expressed at each time level $l = 1, 2, \ldots, m$. The response vector (10) expressed at time level t_j has the following form:

$$\mathbf{q}(t_j) = \sum_k \mathbf{L}_k\,\mathbf{f}_k\,e_k(t_j) - \frac{\Delta t}{T}\sum_{k=-N}^{N} \mathbf{L}_k\,e_k(t_j)\sum_{l=1}^{m} e_{-k}(t_l)\boldsymbol{\xi}\left[\mathbf{q}(t_l)\right], \tag{11}$$

The relation (11) could be written for each time level $j = 1, 2, \ldots, m$. All of these equations correspond to the nonlinear system of algebraic equations which can be written in the following way: Let us introduce vectors

$$\hat{\mathbf{q}} = \begin{bmatrix} \mathbf{q}(t_1) \\ \mathbf{q}(t_2) \\ \vdots \\ \mathbf{q}(t_m) \end{bmatrix} \in \mathbb{R}^{mn,1}, \qquad \hat{\boldsymbol{\xi}}(\hat{\mathbf{q}}) = \begin{bmatrix} \boldsymbol{\xi}\left[\mathbf{q}(t_1)\right] \\ \boldsymbol{\xi}\left[\mathbf{q}(t_2)\right] \\ \vdots \\ \boldsymbol{\xi}\left[\mathbf{q}(t_m)\right] \end{bmatrix} \in \mathbb{R}^{mn,1}. \tag{12}$$

The system of equations (11) can be expressed in the compact form

$$\hat{\mathbf{q}} = \boldsymbol{\Omega}^T\mathbf{L}\,\mathbf{f} - \frac{\Delta t}{T}\boldsymbol{\Omega}^T\mathbf{L}\,\hat{\mathbf{I}}\,\boldsymbol{\Omega}\,\hat{\boldsymbol{\xi}}(\hat{\mathbf{q}}), \tag{13}$$

where

$$\boldsymbol{\Omega} = \begin{bmatrix} e_{-N}(t_1)\mathbf{I} & e_{-N}(t_2)\mathbf{I} & \cdots & e_{-N}(t_m)\mathbf{I} \\ e_{-N+1}(t_1)\mathbf{I} & e_{-N+1}(t_2)\mathbf{I} & \cdots & e_{-N+1}(t_m)\mathbf{I} \\ \vdots & \vdots & \ddots & \vdots \\ e_N(t_1)\mathbf{I} & e_N(t_2)\mathbf{I} & \cdots & e_N(t_m)\mathbf{I} \end{bmatrix} = \left[\mathbf{E}(t_1), \mathbf{E}(t_2), \ldots, \mathbf{E}(t_n)\right] \in \mathbb{C}^{(2N+1)n,nm}, \tag{14}$$

$$\mathbf{I} = \begin{bmatrix} 1 & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{bmatrix} \in \mathbb{R}^{n,n}, \quad \mathbf{E}(t_j) = \begin{bmatrix} e_{-N}(t_j)\mathbf{I} \\ e_{-N+1}(t_j)\mathbf{I} \\ \vdots \\ e_N(t_j)\mathbf{I} \end{bmatrix} \in \mathbb{C}^{(2N+1)n,n}, \tag{15}$$

$$\hat{\mathbf{I}} = \begin{bmatrix} & & & \mathbf{I} \\ & & \iddots & \\ & \mathbf{I} & & \\ \mathbf{I} & & & \end{bmatrix} \in \mathbb{R}^{(2N+1)n,(2N+1)n}, \ \mathbf{L} = \begin{bmatrix} \mathbf{L}_{-N} & & & \\ & \mathbf{L}_{-N+1} & & \\ & & \ddots & \\ & & & \mathbf{L}_N \end{bmatrix} \in \mathbb{C}^{(2N+1)n,(2N+1)n}. \tag{16}$$

Using the Newton's method to solve the nonlinear equation (13), we can come to the iteration rule

$$\hat{\mathbf{q}}_r = \hat{\mathbf{q}}_{r-1} - \lambda \left(\mathbf{I} + \frac{\Delta t}{T} \mathbf{\Omega}^T \mathbf{L} \hat{\mathbf{I}} \mathbf{\Omega} \frac{\partial \hat{\boldsymbol{\xi}}}{\partial \hat{\mathbf{q}}_{r-1}} \right)^{-1} \left(\hat{\mathbf{q}}_{r-1} + \frac{\Delta t}{T} \mathbf{\Omega}^T \mathbf{L} \hat{\mathbf{I}} \mathbf{\Omega} \hat{\boldsymbol{\xi}}(\hat{\mathbf{q}}_{r-1}) - \mathbf{\Omega}^T \mathbf{L} \mathbf{f} \right), \ \lambda \in \langle 0, 1 \rangle, \tag{17}$$

where λ is the coefficient of numerical damping,

$$\frac{\partial \hat{\boldsymbol{\xi}}}{\partial \hat{\mathbf{q}}_{r-1}} = \begin{bmatrix} \frac{\partial \boldsymbol{\xi}}{\partial \mathbf{q}_{r-1}}(t_1) & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \frac{\partial \boldsymbol{\xi}}{\partial \mathbf{q}_{r-1}}(t_2) & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \frac{\partial \boldsymbol{\xi}}{\partial \mathbf{q}_{r-1}}(t_m) \end{bmatrix} \in \mathbb{R}^{nm,nm}, \ \frac{\partial \boldsymbol{\xi}}{\partial \mathbf{q}_{r-1}}(t_j) \in \mathbb{R}^{n,n}, \ j = 1, 2, \ldots, m. \tag{18}$$

Application. The force function has the form $f(t) = \hat{f}(\cos \omega t + 2 \cos 2\omega t + 3 \sin 3\omega t)$, $\hat{f} = 2N$, $\omega = 10\,\mathrm{rad\,s^{-1}}$. Defining a non-smooth nonlinearity (Fig. 2), we introduce the function of unit step $h(q)$. The result of periodic solution and comparison with direct numerical integration is depicted in Fig. 3. The result of numerical continuation (to steady state) is shown in Fig. 4.

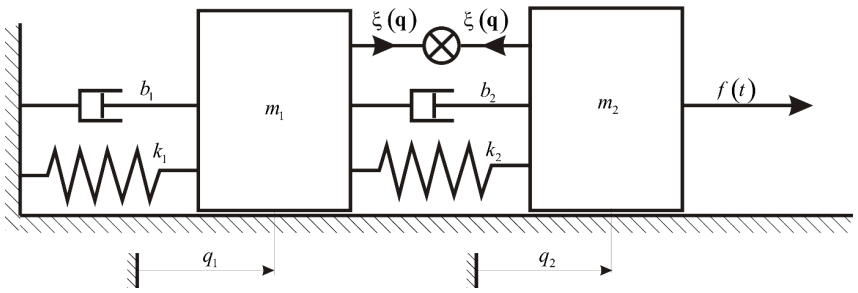

Fig. 1. Nonlinear system with 2 DOF

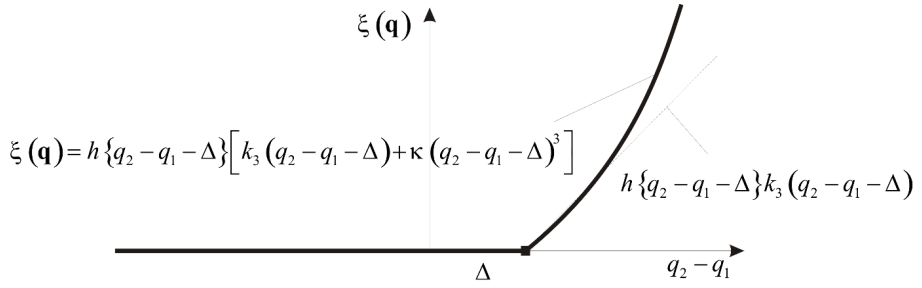

Fig. 2. One side cubic nonlinearity

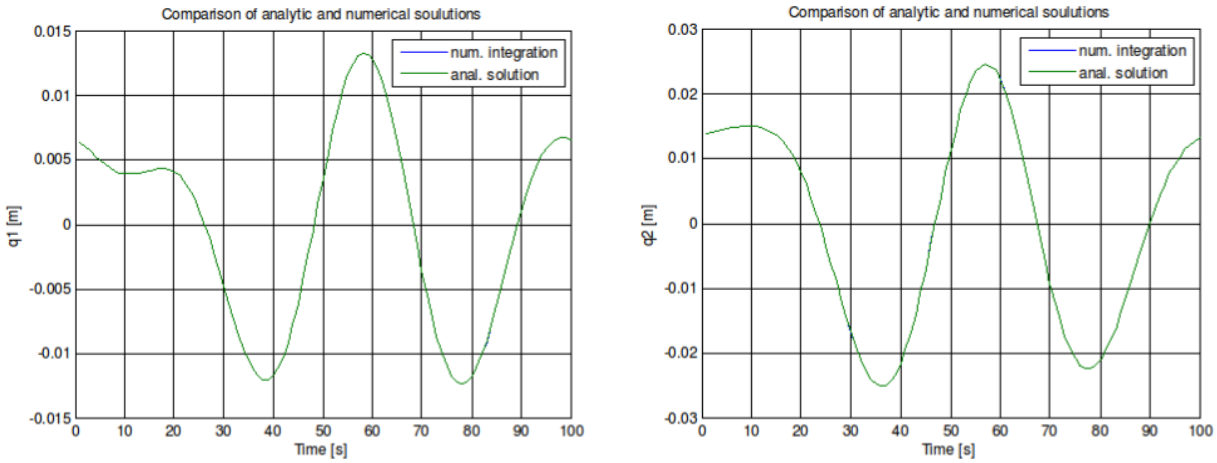

Fig. 3. One period solution comparison of presented and numerical approaches

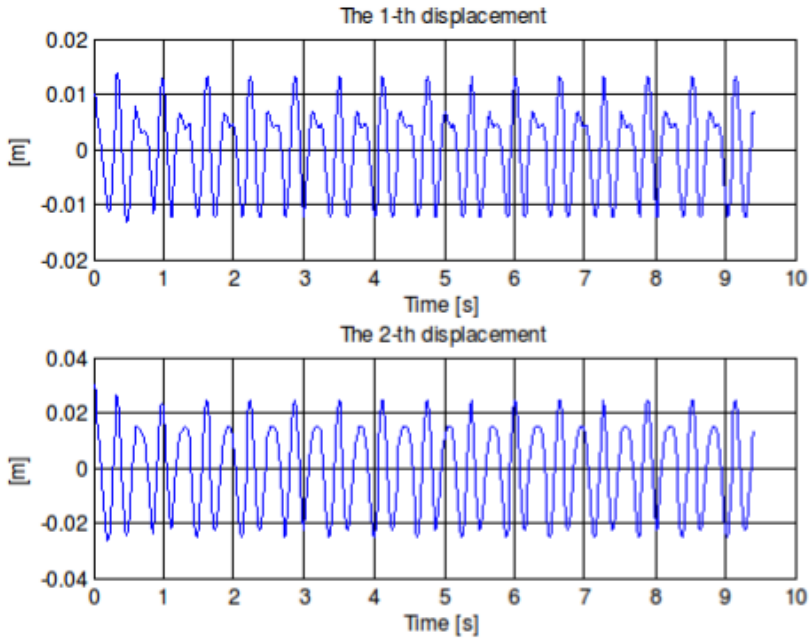

Fig. 4. Numerical continuation

Conclusion. A wide class of nonlinear systems was analysed by means of the presented methodology. We can say that the agreement between the analytical results (presented approach) and numerical integration are on a very good level and the convergence criterion will be subject of analysis in the future.

References

[1] Dupal, J., Periodical solution of n-DOF parametric system vibration, Proceedings of the 37th Conference Computational mechanics, Srní, 2022, pp. 22–23.

[2] Dupal, J., Zajíček, M., Analytical periodic solution and stability assessment of 1 DOF parametric systems with time varying stiffness, Applied Mathematics and Computation 243 (2014) 138–153.