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Comparison and effect of various pedestrian models on a footbridge

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1. Introduction

This paper presents a comparison of selected pedestrian load models and their dynamic effects on a simply supported concrete footbridge. The analysis focuses on a single walking pedestrian and evaluates both stationary and non-stationary responses.

The stationary response is obtained using the Newmark integration method with a time-dependent point force applied at the beam midspan. The non-stationary response is computed using the finite element method (FEM) based on Hermite shape functions. The governing fourth-order differential equation of damped vibrations is solved using time discretization.

2. Pedestrian models

Most pedestrian-induced load models (Fig. 1) can be expressed in the general form of a Fourier series [4]

$$F = G\left[1 + \sum_{i=1}^{n} \alpha_i \sin(2\pi i f t + \phi_i)\right] \quad \text{or} \quad F = G\left[1 + \sum_{i=1}^{n} \alpha_i \cos(2\pi i f t + \phi_i)\right],$$

where G is the pedestrian's weight, α_i is the dynamic coefficient of the i-th harmonic component, f is the step frequency, t is the time, and ϕ_i is the phase angle of the harmonic force.

These models approximate the vertical dynamic force generated by walking, Fig. 2.

3. Newmark integration method

The stationary response is analysed using the Newmark integration method, where a time-dependent point load is applied at the beam midspan to produce the maximum response. The stiffness matrix was obtained by modelling the beam in SCIA Engineer, while the mass matrix was derived by distributing the total mass among discrete points. The damping matrix was

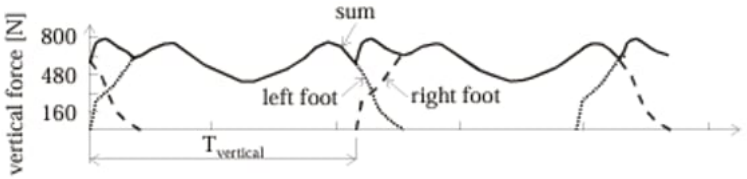

Fig. 1. Example of pedestrian loading functions

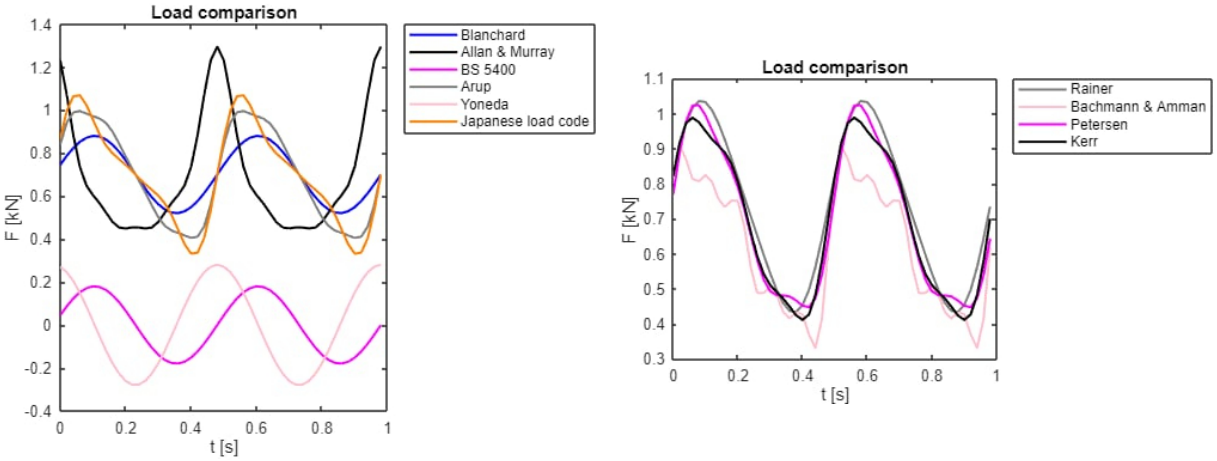

Fig. 2. Comparison of selected pedestrian load models

calculated using the Rayleigh damping model

$$C = \xi\omega_1 M + \frac{\xi}{\omega_1} K, \tag{1}$$

where M is the mass matrix, K is the stiffness matrix, ω_1 is the first natural frequency, and ξ is the damping ratio.

The following coefficients are used in the Newmark integration method:

- average acceleration method: $\alpha = 1/4$, $\delta = 1/2$,
- linear acceleration method: $\alpha = 1/6$, $\delta = 1/2$.

The governing equation for the Newmark method is

$$\left(K + \frac{M}{h^2\alpha} + \frac{C\delta}{h\alpha}\right) u_{n+1} = F_{n+1} + M\left[\frac{\dot{u}_n}{h\alpha} + \frac{u_n}{h^2\alpha} + \frac{\ddot{u}_n(\frac{1}{2} - \alpha)}{\alpha}\right] + \\ + C\left[-\dot{u}_n - \ddot{u}_n h(1-\delta) + \frac{\dot{u}_n\delta}{\alpha} + \frac{u_n\delta}{h\alpha} + \frac{\ddot{u}_n(\frac{1}{2} - \alpha)\delta}{\alpha}\right]. \tag{2}$$

The corresponding expressions for the velocity and acceleration are

$$\dot{u}_{n+1} = \frac{1}{h\alpha}\left[\dot{u}_n h(\alpha - \delta) + \ddot{u}_n h^2(\alpha - \frac{1}{2}\delta) + (u_{n+1} - u_n)\delta\right], \tag{3}$$

$$\ddot{u}_{n+1} = \frac{1}{h^2\alpha}\left[-\dot{u}_n h + u_{n+1} - u_n - \ddot{u}_n h^2\left(\frac{1}{2} - \alpha\right)\right]. \tag{4}$$

4. Finite element method

The finite element method [1–3] allows the load to vary both in time and space. The stiffness and mass matrices are derived using Hermite shape functions ϕ and ψ and their derivatives.

The resulting non-stationary fourth-order differential equation describing damped beam vibrations is

$$\mu\frac{\partial^2 u}{\partial t^2}(x,t) + 2c\frac{\partial u}{\partial t}(x,t) + d\frac{\partial^4 u}{\partial x^4}(x,t) = f(x,t). \tag{5}$$

This equation is solved using the Rothe method for time discretization. For $k \geq 1$, the displacement U_{k+1} is obtained as

$$\frac{U_{k+1} - 2U_k + U_{k-1}}{\tau^2} + c\frac{U_{k+1} - U_{k-1}}{2\tau} + dU''''_{k+1} = f(x, t_k). \tag{6}$$

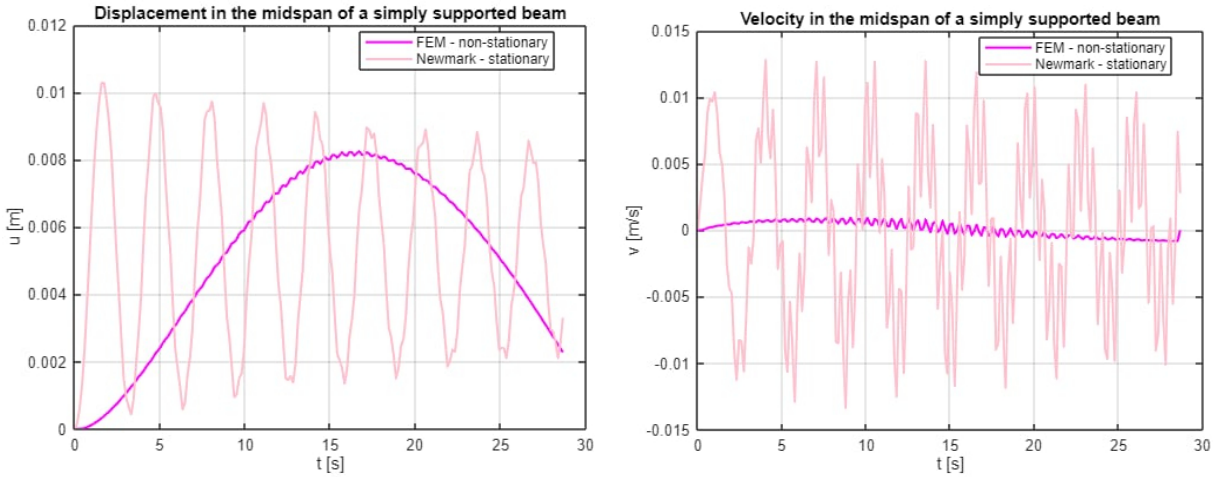

Fig. 3. Comparison of displacement and velocity responses

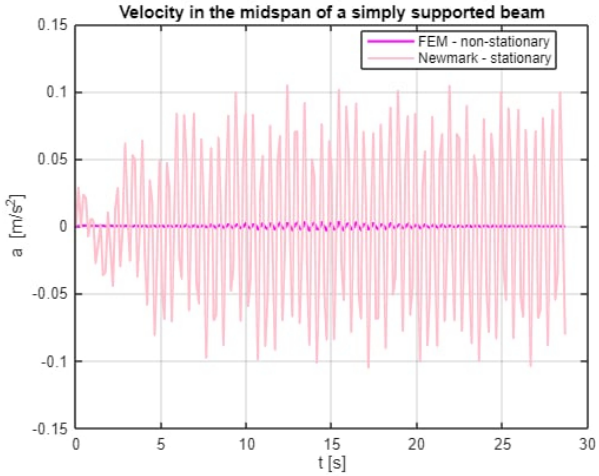

Fig. 4. Comparison of acceleration responses

After spatial discretization using FEM, the system of equations becomes

$$\left[dK + \left(\frac{\mu}{\tau^2} + \frac{c}{\tau}\right) M\right] \vec{\omega}^{(k+1)} = \vec{F}^{(k)} + M \left[\frac{2\mu}{\tau^2}\vec{\omega}^{(k)} + \left(\frac{c}{\tau} - \frac{\mu}{\tau^2}\right) \vec{\omega}^{(k-1)}\right], \tag{7}$$

where d is the bending stiffness (EI), K is the stiffness matrix, μ is the linear mass density, τ is the time step, c is the damping coefficient, M is the mass matrix, and F is the applied force.

5. Comparison

Comparison of stationary and non-stationary responses using the Blanchard's pedestrian model is shown in Figs. 3–4.

6. Conclusion

The finite element method (FEM) provides higher numerical accuracy and allows for theoretical verification of convergence using an *a priori* error estimate. The accuracy is assessed by computing the error between the numerical and exact analytical solution as

$$e(x, t_k) = U_k(x) - \hat{u}(x, t_k),$$

where $U_k(x)$ is the FEM approximation and $\hat{u}(x, t_k)$ is the exact solution.

The L^2-norm of the error is then evaluated as

$$\|e(x,t_k)\|^2_{L^2(0,1)} = \vec{e}_k^T \mathbf{M}\, \vec{e}_k,$$

where $\mathbf{M}$ is the mass matrix, and $\vec{e}_k$ is the vector of nodal errors. The theoretical convergence behaviour satisfies the relation

$$\max_k \|e(x,t_k)\|_{L^2(0,1)} \leq c\,(\tau^2 + h^4),$$

confirming the expected dependence of the error on both the temporal step size τ and the spatial discretization h.

Although FEM yields a higher precision and allows for theoretical error verification, it is more complex to implement, particularly for geometrically intricate structures. The Newmark integration method, on the other hand, produces slightly larger response values but remains computationally simpler, making it a practical and safe option for structural design applications.

References

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